International Journal of Bifurcation and Chaos, Vol. 9, No. 8 (1999) 1619–1634

APPLICATION OF ORDER PARAMETER EQUATIONS FOR THE ANALYSIS AND THE CONTROL OF NONLINEAR TIME DISCRETE DYNAMICAL SYSTEMS

P. LEVI, M. SCHANZ, S. KORNIENKO* and O. KORNIENKO
Institute of Parallel and Distributed High-Performance Systems, University of Stuttgart, Breitwiesenstr. 20-22, D-70565 Stuttgart, Germany

Received January 19, 1999; Revised March 4, 1999

This work is based on the concept of order parameters of synergetics. The order parameter equations describe the behavior of a system in the vicinity of an instability and are used here not only for the analysis but also for the control of nonlinear time discrete dynamical systems. Usually, the dimensionality of the evolution equations of the order parameters is less than the dimensionality of the original evolution equations. It is, therefore, convenient to introduce control mechanisms, first in the order parameter equations, and then to use the obtained results for the control of the original system. The aim of the control in this case is to avoid chaotic behavior of the system. This is achieved by shifting appropriate bifurcation points of a period-doubling cascade. In this work we concentrate on the shifting of only the first bifurcation point. The used control mechanisms are delayed feedback schemes. As an example the well-known Hénon map is investigated. The order parameter equation is calculated using both the adiabatic elimination procedure and the center manifold theory. Using the order parameter concept two types of control mechanisms are constructed, analyzed and compared.

1. Introduction

The analysis of nonlinear dynamical systems and the control of the chaotic behavior often occurring in such systems is very important. In many cases, for example in technical systems, chaotic behavior may lead to failures, overloads or even damages and therefore has to be strictly avoided. Many publications (see, e.g. [Ott *et al.*, 1990; Pyragas, 1993; Wang & Abed, 1995; Alvarez-Ramírez, 1993; Friedel *et al.*, 1997]) are devoted to this topic. There are some differences between these works and the article presented here. First, the calculation of the parameters for the control is done analytically and is based on the concept of order parameters of synergetics [Haken, 1983] of nonlinear dynamical systems. Second, we do not stabilize an unstable periodic orbit of a strange attractor, but extend the working area in the parameter space by shifting appropriate bifurcation points as suggested in [Pyragas, 1993]. The concept of order parameters allows on the one hand the analysis of the system in the vicinity of an instability, i.e. a bifurcation point, and on the other hand the development of a general control mechanism, which makes it possible to change the behavior of the system in an appropriate way. In this article we focus on the investigation and the control of the period-doubling bifurcations in time discrete dynamical systems. The control, i.e. the shift of a specific bifurcation point in the period-doubling cascade, is achieved by applying a specific delayed feedback scheme to the original

*E-mail: korniesi@informatik.uni-stuttgart.de

system. The advantage of this control mechanism is that it can be applied without any *a piori* knowledge about the system which is needed for instance if one uses the well-known method suggested by [Ott *et al.*, 1990] to control the dynamical behavior of the system.

The concept of the presented general control mechanism consists of the following: The order parameters describe the behavior of the nonlinear dynamical system in the vicinity of an instability. Using delayed feedback functions (see, e.g. [Pyragas, 1993]) in the order parameter equations, it is possible to change their behavior in an intended way. The advantage of introducing the control mechanism in the order parameter equations is that their dimensionality is less than the dimensionality of the initial evolution equations. Consequently, the calculations become considerably simpler. A synergetic analysis of the order parameter equation with control leads finally to an efficient method for the insertion of control mechanisms into the original system. As an example we investigate the well-known Hénon map [Hénon, 1976].

The article is structured as follows: In Sec. 2 a synergetic analysis of the Hénon system and the derivation of the center manifold and the order parameter equation is shown. In Sec. 3 the parameters for the control of the system based on delayed feedback schemes are calculated. Finally, two different control methods with delayed feedback are considered and compared.

2. Synergetic Analysis of Time Discrete Nonlinear Dynamical Systems

2.1. *The synergetic analysis approach*

The main objective, which we pursue in this section, consists of the derivation of low-dimensional evolution equations, which describe the behavior of a system near an instability. In terms of synergetics, such equations are called order parameter equations [Haken, 1983, 1988]. It is assumed, that the state vector $\underline{q}_n$ of the dynamical system of interest satisfies the following time discrete nonlinear autonomous equation of motion

$$\underline{q}_{n+1} = \underline{N}(\underline{q}_n, \{\alpha\}), \tag{1}$$

where $\underline{N}$ is a nonlinear vector function, and $\{\alpha\}$ a set of control parameters. The stationary states of the system are then defined by

$$\underline{q}_{st} = \underline{N}(\underline{q}_{st}, \{\alpha\}). \tag{2}$$

Our further task consists of the analysis of small deviations from the stationary state $\underline{q}_{st}$. In the vicinity of an instability we can decompose the state vector of the system as follows [Haken, 1983]:

$$\underline{q}_n = \underline{q}_{st} + \underline{\Delta q}_n = \underline{q}_{st} + \sum_i \xi_n^i \underline{v}_i\,, \tag{3}$$

where $\underline{v}_i$ are the eigenvectors of the Jacobian of the system (1), while ξ_n^i are time-dependent coefficients. Substituting Eq. (3) in Eq. (1) and using the time-scale hierarchy

$$|\lambda_u| \geq 1, \quad |\lambda_s| < 1\,, \tag{4}$$

where λ_u, λ_s (s — stable, u — unstable) are the eigenvalues of the Jacobian of the system (1), evaluated at the stationary state $\underline{q}_{st}$, we distinguish the linear modes $\underline{v}_i$ and finally arrive at the following mode equations in the case of a two-dimensional state space:

$$\xi_{n+1}^u = \lambda_u \xi_n^u + N_u(\xi_n^u, \xi_n^s, \{\alpha\}) \tag{5a}$$

$$\xi_{n+1}^s = \lambda_s \xi_n^s + N_s(\xi_n^u, \xi_n^s, \{\alpha\})\,, \tag{5b}$$

where ξ_n^u, ξ_n^s are the amplitudes of the unstable and stable modes and N_u, N_s are nonlinear functions of ξ_n^u, ξ_n^s. Using the slaving principle, (see, e.g. [Haken, 1983]) we can express the stable mode amplitude as function of the unstable mode amplitude via the center manifold or adiabatic elimination procedure

$$\xi_n^s = h(\xi_n^u)\,. \tag{6}$$

Substituting Eq. (6) in Eq. (5b) we obtain an implicit equation for the center manifold $h(\xi_n^u)$

$$\begin{aligned} h(\xi_{n+1}^u) &= h(\lambda_u \xi_n^u + N_u(\xi_n^u, h(\xi_n^u), \{\alpha\})) \\ &= \lambda_s h(\xi_n^u) + N_s(\xi_n^u, h(\xi_n^u), \{\alpha\})\,. \end{aligned} \tag{7}$$

Substituting Eq. (6) in Eq. (5a) we obtain the order parameter equation

$$\xi_{n+1}^u = \lambda_u \xi_n^u + N_u(\xi_n^u, h(\xi_n^u), \{\alpha\})\,, \tag{8}$$

which describes the behavior of the system in the neighborhood of the instability and no longer depends on the stable mode amplitude ξ_n^s. For simplification we denote further the mode amplitudes

ξ_n^s, ξ_n^u, often only as modes and the corresponding mode amplitude equations as mode equations.

2.2. *Calculation of the order parameter equation of the Hénon map*

2.2.1. *Linear stability analysis*

To demonstrate this approach we will treat the well-known two-dimensional Hénon map [Hénon, 1976] as an example, because its behavior has been investigated thoroughly so that we are able to compare our result with those already known and presented in, for instance [Giovannozzi, 1993; Alessandro *et al.*, 1990].

The Hénon map is defined by the following set of equations:

$$\begin{aligned} x_{n+1} &= 1 + y_n - a x_n^2 \\ y_{n+1} &= b x_n\,. \end{aligned} \tag{9}$$

We have restricted our investigations to the parameter $b = 0.3$ and focus on the first bifurcation of the period-doubling cascade which occur at the critical parameter value $a_{\mathrm{crit}} = 147/400 = 0.3675$. For this bifurcation we derive the order parameter equation in the above formulated straight forward way and analyze its properties. The state vector of this system is defined by $\underline{q}_n = (q_n^1,\, q_n^2)^T = (x_n,\, y_n)^T$. The stationary states $\underline{q}_{st}$ of Eq. (9) are given by

$$\begin{aligned} x_{st}^{1,2} &= \frac{b - 1 \pm \sqrt{1 - 2b + b^2 + 4a}}{2a} \\ y_{st}^{1,2} &= \frac{b(b - 1 \pm \sqrt{1 - 2b + b^2 + 4a})}{2a} \end{aligned} \tag{10}$$

and the Jacobian of the system (9) evaluated at the stationary states (10) is then

$$\underline{\underline{J}} = \begin{pmatrix} -2a x_{st} & 1 \\ b & 0 \end{pmatrix}. \tag{11}$$

For the eigenvalues of the matrix (11) we get

$$\lambda_{1,2} = -a x_{st} \pm \sqrt{a^2 x_{st}^2 + b}\,, \tag{12}$$

and the corresponding eigenvectors are:

$$\underline{v}_1 = \begin{pmatrix} \dfrac{\lambda_1}{b} \\ 1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} \dfrac{\lambda_2}{b} \\ 1 \end{pmatrix}. \tag{13}$$

From the linear stability analysis of the system it follows that the stationary state $(x_{st}^2,\, y_{st}^2)$ is unstable because the condition for a contracting linear map $|\lambda| < 1$ is violated in the whole region of interest of the parameter a, i.e. the interval $[0, 1.4]$. The stationary state $(x_{st}^1,\, y_{st}^1)$ loses its stability at the parameter value $a_{\mathrm{crit}} = 0.3675$, because at this value of the parameter a the condition for a contracting linear map $|\lambda| < 1$ is violated by the eigenvalue λ_2. As a consequence we focus on the following for the stable stationary state $(x_{st}^1,\, y_{st}^1)$ and denote the corresponding eigenvalue λ_2 from now on as λ_u and the other eigenvalue λ_1 as λ_s.

2.2.2. *Calculation of the mode amplitude equations*

In the following we use the compact tensor notation for our investigations

$$\begin{aligned} \underline{q}_{n+1} &= \Gamma_{(1)} + \Gamma_{(2)}(:\underline{q}_n) + \Gamma_{(3)}(:\underline{q}_n)^2 \\ &\quad + \Gamma_{(4)}(:\underline{q}_n)^3 \ldots = \sum_{r=0}^{p} \Gamma_{(r+1)}(:\underline{q}_n)^r\,. \end{aligned} \tag{14}$$

Expression (14) reads in components:

$$\begin{aligned} q_{n+1}^i &= (\Gamma_{(1)})_i + \sum_{j_1} (\Gamma_{(2)})_{ij_1} q_n^{j_1} \\ &\quad + \sum_{j_1 j_2} (\Gamma_{(3)})_{ij_1 j_2} q_n^{j_1} q_n^{j_2} + \cdots \\ &= \sum_{r=0}^{p} \sum_{ij_1\cdots j_r} (\Gamma_{(r+1)})_{ij_1\cdots j_r} q_n^{j_1} \cdots q_n^{j_r}\,. \end{aligned} \tag{15}$$

Herein p is the order of the equation of motion and the indices in brackets denote the rank of the corresponding tensors. The nonvanishing components of the tensors for the Hénon map (9) are shown in Table 1. For further analysis we investigate the behavior of small deviations from the stationary states. Therefore we insert $\underline{q}_n = \underline{q}_{st} + \underline{\Delta q}_n = (x_{st} + \Delta x_n,\, y_{st} + \Delta y_n)$ in the evolution Eq. (14)

Table 1. Nonvanishing tensor components $\Gamma_{(1)}$, $\Gamma_{(2)}$, $\Gamma_{(3)}$ of the Hénon map.

$\Gamma_{(1)}$	$\Gamma_{(2)}$	$\Gamma_{(3)}$
$(\Gamma_{(1)})_1 = 1$	$(\Gamma_{(2)})_{12} = 1$	$(\Gamma_{(3)})_{111} = -a$
$(\Gamma_{(1)})_2 = 0$	$(\Gamma_{(2)})_{21} = b$	

Table 2. Nonvanishing tensor components $\Gamma^L_{(2)}$, $\Gamma^N_{(3)}$ of the Hénon map.

$\Gamma^L_{(2)}$	$\Gamma^N_{(3)}$
$(\Gamma^L_{(2)})_{11} = -2ax_{st}$	$(\Gamma^N_{(3)})_{111} = -a$
$(\Gamma^L_{(2)})_{12} = 1$	
$(\Gamma^L_{(2)})_{21} = b$	

and derive the equation of motion for the deviation $\underline{\Delta q}_n$

$$\underline{\Delta q}_{n+1} = \Gamma^L_{(2)}(:\underline{\Delta q}_n) + \sum_{r=2}^{p} \Gamma^N_{(r+1)}(:\underline{\Delta q}_n)^r\,, \quad (16)$$

where Γ^L represents the linear part which is given by the Jacobian and Γ^N represent the nonlinear parts. The nonvanishing components of the tensors in Eq. (16) for the systems (9) are shown in Table 2. To proceed further we apply the following coordinate transformation introducing hereby the modes ξ^s_n, ξ^u_n and $\xi_n = (\xi^s_n, \xi^u_n)^T$:

$$\underline{\Delta q}_n = \sum_k \xi^k_n \underline{v}_k = \underline{\underline{V}}\,\underline{\xi}_n = \Gamma^V_{(2)}(:\underline{\xi}_n)\,, \quad (17)$$

where the columns of the matrix $\underline{\underline{V}}$ are given by the eigenvectors of the Jacobian, i.e. $\underline{\underline{V}} = (\underline{v}^1, \underline{v}^2)$ and $\Gamma^V_{(2)}$ is the corresponding tensor. Inserting Eq. (17) in Eq. (16) we get then

$$\Gamma^V_{(2)}(:\underline{\xi}_{n+1}) = \Gamma^L_{(2)}\Gamma^V_{(2)}(:\underline{\xi}_n) + \Gamma^N_{(3)}(\Gamma^V_{(2)}(:\underline{\xi}_n))^2\,. \quad (18)$$

Multiplying with the inverse tensor $\Gamma^{V^{-1}}_{(2)}$ from the left side we obtain the mode equations

$$\underline{\xi}_{n+1} = \Gamma^\Lambda_{(2)}(:\underline{\xi}_n) + \Gamma^{\tilde{N}}_{(3)}(:\underline{\xi}_n)^2 \quad (19)$$

with the tensor products $\Gamma^\Lambda_{(2)} = \Gamma^{V^{-1}}_{(2)}\Gamma^L_{(2)}\Gamma^V_{(2)}$ and $\Gamma^{\tilde{N}}_{(3)} = \Gamma^{V^{-1}}_{(2)}\Gamma^N_{(3)}\Gamma^V_{(2)}\Gamma^V_{(2)}$. We thereby make use of the identity

$$(\Gamma^{V^{-1}}_{(2)}\Gamma^V_{(2)})_{ij} = \delta_{ij}\,, \quad (20)$$

where δ_{ij} is the well-known Kronecker-Symbol. As an example we show the explicit calculation of the tensor

$$\Gamma^{\tilde{N}}_{(3)} = \Gamma^{V^{-1}}_{(2)}\Gamma^N_{(3)}\Gamma^V_{(2)}\Gamma^V_{(2)}\,, \quad (21)$$

which reads in components:

$$(\Gamma^{\tilde{N}}_{(3)})_{ijk} = \sum_{lmn} (\Gamma^{V^{-1}}_{(2)})_{il}(\Gamma^N_{(3)})_{lmn}(\Gamma^V_{(2)})_{mj}(\Gamma^V_{(2)})_{nk}\,. \quad (22)$$

The nonvanishing components of the tensor $\Gamma^\Lambda_{(2)}$ are $(\Gamma^\Lambda_{(2)})_{11} = \lambda_s$ and $(\Gamma^\Lambda_{(2)})_{22} = \lambda_u$. With respect to the tensor components, summarized in Table 2, Eq. (19) reads in detail (we denote as ξ^u_n the mode with eigenvalue λ_u and ξ^s_n the mode with the eigenvalue λ_s):

$$\xi^s_{n+1} = \lambda_s\xi^s_n - \frac{a(\lambda_s\xi^s_n + \lambda_u\xi^u_n)^2}{b(\lambda_s - \lambda_u)} \quad (23a)$$

$$\xi^u_{n+1} = \lambda_u\xi^u_n + \frac{a(\lambda_s\xi^s_n + \lambda_u\xi^u_n)^2}{b(\lambda_s - \lambda_u)}\,, \quad (23b)$$

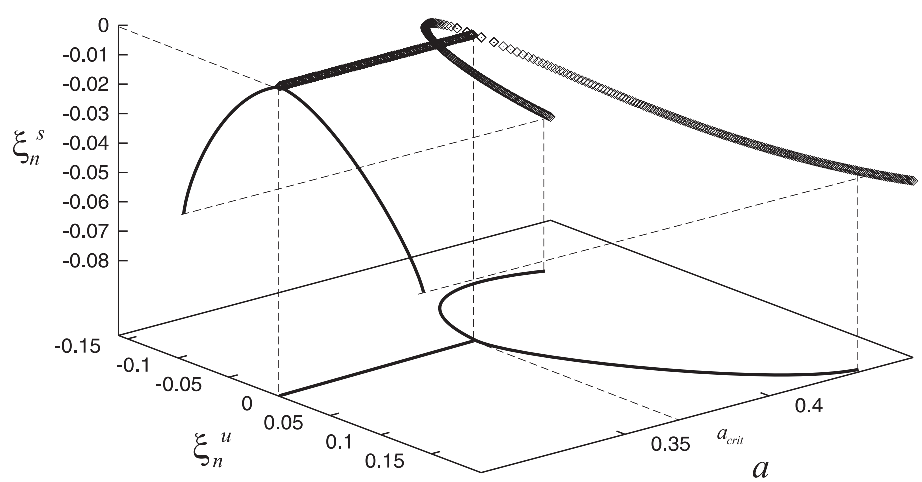

Fig. 1. The bifurcation diagram of the mode equations (23) at the first bifurcation in a 3D representation.

where the first equation is the one for the stable mode and the second one is for the unstable mode. Equations (23) are the mode equations of the system (9). They describe the behavior of the stable and unstable modes of the Hénon map at the first bifurcation of the period-doubling cascade (see Figs. 1 and 10). In terms of synergetics the unstable mode ξ_n^u is the order parameter, while the stable mode ξ_n^s corresponds to the enslaved mode, as was already mentioned in Sec. 2.1.

2.2.3. *Calculation of the order parameter equation using the adiabatic approximation*

Near an instability the stable mode ξ_n^s shows an interesting characteristic behavior. The dynamics is mainly influenced by two effects: first an intrinsic dynamics which is governed by the eigenvalue λ_s and, second an external dynamics which is governed by the nonlinear coupling with the unstable mode. Due to the time-scale hierarchy expressed in (4) the intrinsic dynamic of the stable mode ξ_n^s is much faster than that of the unstable mode ξ_n^u. Therefore one can say that the intrinsic dynamics decays very fast and only the dynamics which is caused by the unstable mode remains. This characteristic behavior is mathematically expressed by the center manifold theorem. To derive an approximation of the center manifold $h(\xi_n^u)$ we can exploit this behavior using the adiabatic approximation procedure for the stable mode

$$\xi_{n+1}^s = \xi_n^s \,. \tag{24}$$

After this calculation we can determine, whether the properties:

$$h(0) = 0\,, \tag{25}$$

$$\left.\frac{dh(\xi_n^u)}{d\xi_n^u}\right|_{\xi_n^u=0} = 0 \tag{26}$$

are fulfilled. From Eq. (23a) we derive the following equation for the stable mode ξ_n^s:

$$\xi_n^s = \lambda_s\xi_n^s - \frac{a(\lambda_s\xi_n^s + \lambda_u\xi_n^u)^2}{b(\lambda_s - \lambda_u)}\,. \tag{27}$$

Solving this quadratic equation we obtain finally the following approximation of the center manifold:

$$\xi_n^{s_1,s_2} = \frac{2ab\xi_n^u - K_a \pm \sqrt{K_a(K_a - 4ab\xi_n^u)}}{2a\lambda_s^2} \tag{28}$$

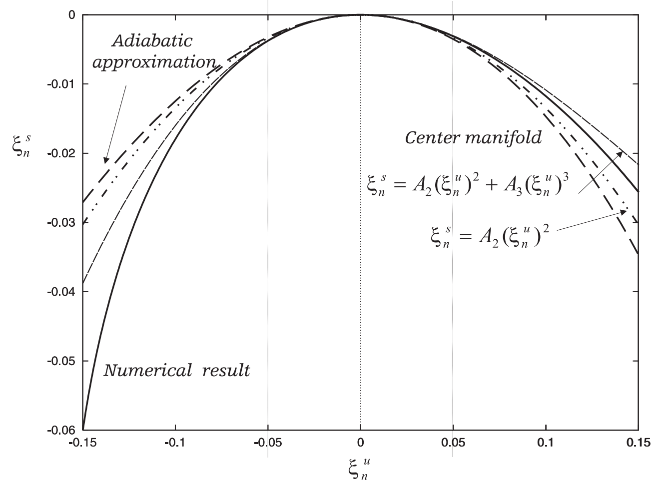

Fig. 2. Dependence between the variables ξ_n^u, ξ_n^s, obtained by a simulation and analytically, using the adiabatic approximation (28) and the center manifold (31).

where

$$K_a = b(\lambda_s - 1)(\lambda_u - \lambda_s) \tag{29}$$

and $\lambda_s\lambda_u = -b$. In Eq. (28) we select the sign of the square root so that the order parameter equation for values $a < 0.3675$ lead to the already known stationary state (x_{st}^1, y_{st}^1). This condition is satisfied by the plus sign in Eq. (28). Substituting the value $\xi_n^u = 0$ in the expression for $\xi_n^{s_1} = h(\xi_n^u)$ we immediately obtain $\xi_n^s = 0$, which means that requirement (25) is fulfilled. Similarly calculating the derivative of $\xi_n^{s_1}$ with respect to ξ_n^u and substituting again $\xi_n^u = 0$ we see that (26) is also fulfilled. The dependence of $\xi_n^{s_1} = \xi_n^s = h(\xi_n^u)$ on the unstable mode ξ_n^u is shown in Fig. 2.

Substituting the expression for the center manifold in that part of Eqs. (23) which represents the unstable mode we yield the order parameter equation for the first bifurcation via the adiabatic approximation procedure

$$\xi_n^u = \lambda_u\xi_n^u + \frac{a}{b(\lambda_s - \lambda_u)} \times \left(\frac{2ab\xi_n^u - K_a + \sqrt{K_a(K_a - 4ab\xi_n^u)}}{2a\lambda_s} + \lambda_u\xi_n^u\right)^2. \tag{30}$$

where K_a is defined by the expression (29).

2.2.4. *Calculation of the order parameter equation using the center manifold theorem*

In this section we show the calculation of the order parameter equation, using the well-known

Table 3. Expansion of the terms of Eqs. (34) into a Taylor series with respect to ε up to the relevant order.

Terms	Taylor Series
$\lambda_u(a_{\text{crit}}+\varepsilon)$	$\lambda_u^0+\lambda_u^1\varepsilon+\cdots$
$\lambda_s(a_{\text{crit}}+\varepsilon)$	$\lambda_s^0+\lambda_s^1\varepsilon+\cdots$
$K=\dfrac{(a_{\text{crit}}+\varepsilon)}{(b(\lambda_s(a_{\text{crit}}+\varepsilon)-\lambda_u(a_{\text{crit}}+\varepsilon))}$	$K^0+\cdots$
$\Phi=(\lambda_s(a_{\text{crit}}+\varepsilon)(A_2\varepsilon^{2\alpha}+A_3\varepsilon^{3\alpha})+\lambda_u(a_{\text{crit}}+\varepsilon)\varepsilon^{\alpha})^2$	$2A_2\lambda_s^0\lambda_u^0\varepsilon^{3\alpha}+(\lambda_u^0)^2\varepsilon^{2\alpha}+\cdots$

Table 4. Coefficient α for Eqs. (43), (57), (68) and (86).

Control	Order Param. Eq. (43)	Mode Eqs. (57)	Mode Eqs. (68)	Eqs. (86)
$a_{\text{crit}}=0.3675$, $k=0$	0.500684	0.500666	0.500666	0.498603
$(a=0.36751-0.36752)$	0.499234	0.499215	0.499215	0.501297
$a_{\text{crit}}=1$, $k=-0.302893182$	0.500703	0.500624	0.500625	0.499094
$(a=1.00001-1.00002)$	0.499247	0.499330	0.499330	0.500871
$a_{\text{crit}}=1.4$, $k=-0.461869429$	0.500637	0.500572	0.500573	0.499216
$(a=1.40001-1.40002)$	0.499312	0.499379	0.499380	0.500745

center manifold theorem [Carr, 1981; Wunderlin, 1981; Wiggins, 1990]. Considering Eq. (6), we use the following supposition about a center manifold:

$$\xi_n^s = h(\xi_n^u) = A_2(\xi_n^u)^2 + A_3(\xi_n^u)^3 + O(4)\,, \tag{31}$$

$$\xi_{n+1}^s = h(\xi_{n+1}^u) = A_2(\xi_{n+1}^u)^2 + A_3(\xi_{n+1}^u)^3 + O(4)\,, \tag{32}$$

where the coefficients A_2, A_3 are determined from the solution of Eq. (7). For the calculations, we use the following asymptotic assumption:

$$\xi_n^u = \beta\varepsilon^{\alpha} + O(\varepsilon^{2\alpha}) \tag{33}$$

which will be confirmed numerically (see Table 4). Here $\varepsilon = a - a_{\text{crit}}$ is the so-called smallness parameter, $\alpha = 1/2$ and a_{crit} is the critical value of the bifurcation parameter a. Our further approach is stated in the following: Using $a = \varepsilon + a_{\text{crit}}$ we expand all ε-dependent expressions $a, \lambda_u(a), \lambda_s(a)$ in Eqs. (23) into a Taylor series with respect to ε and take into account only terms up to the order of $\varepsilon^{3\alpha}$. Because $\varepsilon^{3\alpha} = \varepsilon^{\frac{3}{2}}$ we do not consider terms with the order $\varepsilon^{2\alpha+1} = \varepsilon^{\frac{4}{2}}$.

First we rewrite Eqs. (23) in the following form:

$$\begin{aligned}\xi_{n+1}^s &= \lambda_s\xi_n^s - K\Phi\\ \xi_{n+1}^u &= \lambda_u\xi_n^u + K\Phi\,,\end{aligned} \tag{34}$$

with the abbreviations:

$$K = \frac{a}{b(\lambda_s - \lambda_u)}, \quad \Phi = (\lambda_s\xi_n^s + \lambda_u\xi_n^u)^2\,. \tag{35}$$

Using this approach the resulting terms of Eqs. (34) are summarized in Table 3. From now on whenever in an expression an upper index exist which is equal to zero we denote with it the lowest order of this expression. Therefore the expressions in Table 3 are:

$$K^0 = \frac{a_{\text{crit}}}{b(\lambda_s^0 - \lambda_u^0)}, \quad \lambda_u^0 = \lambda_u(a_{\text{crit}}), \quad \lambda_s^0 = \lambda_s(a_{\text{crit}}).$$

Using Eq. (7) for the calculation of the center manifold we yield in this case

$$h(\xi_{n+1}^u) = \lambda_s h(\xi_n^u) - K\Phi\,.$$

where K, Φ are given by (35). Obeying further Eqs. (31)–(35) and keeping only terms up to the order $\varepsilon^{3\alpha}$ we end up with

$$A_2\left(\lambda_u^0\varepsilon^\alpha + \frac{a_{\text{crit}}}{b(\lambda_s^0-\lambda_u^0)}(\lambda_u^0)_\beta^2\varepsilon^{2\alpha}\right)^2 + A_3(\lambda_{u\beta}^0\varepsilon^\alpha)^3$$
$$= \lambda_s^0(A_2\varepsilon^{2\alpha} + A_{3\beta}\varepsilon^{3\alpha})$$
$$-\frac{a_{\text{crit}}}{b(\lambda_s^0-\lambda_u^0)}(2A_2\lambda_s^0\lambda_{u\beta}^0\varepsilon^{3\alpha} + (\lambda_u^0)^2\varepsilon^{2\alpha})\,. \quad (36)$$

From this equation we determine the coefficients A_2, A_3:

$$A_2 = \frac{a_{\text{crit}}}{b(\lambda_s^0-\lambda_u^0)}\frac{(\lambda_u^0)^2}{(\lambda_s^0-(\lambda_u^0)^2)}\,, \quad (37)$$

$$A_3 = 2A_2\frac{a_{\text{crit}}}{b(\lambda_s^0-\lambda_u^0)}\frac{\lambda_u^0(\lambda_s^0+(\lambda_u^0)^2)}{(\lambda_s^0-(\lambda_u^0)^3)}\,. \quad (38)$$

Thus we yield the cubic center manifold (31) in the following form:

$$\xi_n^s = \frac{a_{\text{crit}}}{b(\lambda_s^0-\lambda_u^0)}\frac{(\lambda_u^0)^2}{(\lambda_s^0-(\lambda_u^0)^2)}$$
$$\times\left((\xi_n^u)^2 + 2\frac{a_{\text{crit}}}{b(\lambda_s^0-\lambda_u^0)}\frac{\lambda_u^0(\lambda_s^0+(\lambda_u^0)^2)}{(\lambda_s^0-(\lambda_u^0)^3)}(\xi_n^u)^3\right)\,. \quad (39)$$

The projection of the numerically obtained bifurcation diagram of the mode equations (23) (see Fig. 1) on the plane ξ_n^u, ξ_n^s show the dependence between the variables ξ_n^u and ξ_n^s. This dependence should coincide with that of Eq. (39) at least in the vicinity of $\xi_n^u = 0$. The dependence between the variables ξ_n^u, ξ_n^s, obtained numerically and analytically are shown in Fig. 2. Substituting Eq. (39) in Eq. (23b) we yield the order parameter equation

$$\xi_{n+1}^u = \lambda_u\xi_n^u + \frac{a}{b(\lambda_s-\lambda_u)}$$
$$\times(\lambda_s A_2(\xi_n^u)^2 + \lambda_s A_3(\xi_n^u)^3 + \lambda_u\xi_n^u)^2\,, \quad (40)$$

where the coefficients A_2, A_3 are given by the expressions (37) and (38). The bifurcation diagrams of ξ_n^u with respect to the parameter a of the mode equations, the adiabatic approximation (30) and the order parameter equation with a quadratic and a cubic center manifold are shown in Fig. 3. Numerical simulations of Eq. (40) show, that our assumption $\xi_n^u \sim \varepsilon^{\frac{1}{2}}$ in the vicinity of the bifurcation point is satisfied (see Table 4).

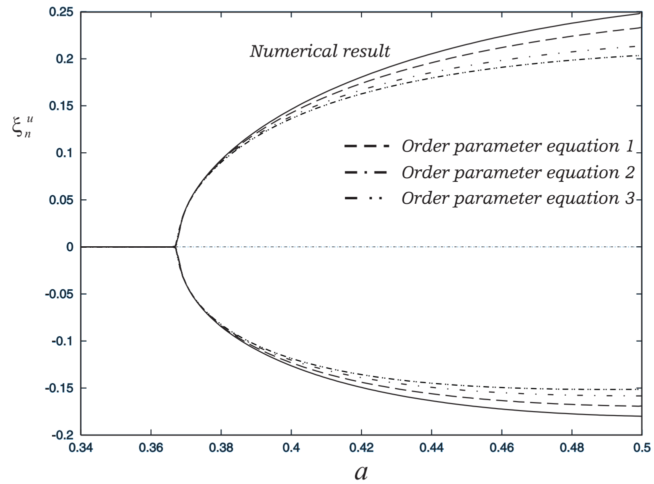

Fig. 3. Bifurcation diagram of the unstable mode of the mode equations (23) and the order parameter equation (40) with a quadratic and cubic center manifold. The order parameter 1 is obtained with the cubic center manifold, the order parameter 2 — with the quadratic center manifold and the order parameter 3 is based on the adiabatic elimination procedure.

3. Control Mechanisms

Controlling the behavior of a dynamical system is of great practical interest, because it is often the case that the behavior is not suitable for a given purpose. So the question arises how one can implement in a given dynamical system control mechanisms which guarantee a suitable behavior. Our idea in this context is to exploit the knowledge obtained by the investigation of the order parameters of the system. Although they are only defined in the neighborhood of an instability this is not a crucial restriction, because the controlling of a system at an instability is required in order to avoid the instability which leads to unsuitable behavior. Thus it is expected, that one can use the fact that the number of degrees of freedom in the order parameter equation is reduced, compared to that with the original system. Our intention here is to proceed in two steps. In the first step we introduce suitable control mechanisms only in the order parameter equation and investigate its stability in order to validate the control mechanism. Whenever the control mechanism is found to be suitable enough, our second step is to then implement exactly the same control mechanism into the mode equations. This is due to the fact that it is not possible to derive the mode equations only from the knowledge of the order parameter equation by itself. After the insertion of the control mechanism into the mode equations it is possible to calculate the corresponding original equations. To demonstrate the usefulness of this approach we have selected a control mechanism which was suggested first by Pyragas [1992, 1993], and which uses feedback functions with delay of the following form:

$$F = k(x_{n-1} - x_n)\,, \tag{41}$$

where k is a parameter. With these functions, we have attempted to avoid high periodic and even chaotic behavior at least in a confined region in the parameter space which we call "working area" by shifting the bifurcation point in an appropriate way. Hereby all other bifurcations in the scenario will be shifted also although not by the same value. This is due to the scaling properties of the bifurcation diagram.

3.1. *Controlling the order parameter equation*

The simplest solution regarding the shift of a bifurcation point is the insertion of a parameter k only in the linear part of the order parameter equation (40) with the quadratic center manifold

$$\xi^u_{n+1} = (\lambda_u - k)\xi^u_n + \frac{a}{b(\lambda_s - \lambda_u)}(\lambda_s A_2(\xi^u_n)^2 + \lambda_u \xi^u_n)^2\,. \tag{42}$$

where A_2 is defined by (37). This solution however is trivial and it is usually not possible to considerably shift the bifurcation point with this method due to the effect of the nonlinear part of (42).

Introducing the feedback functions (41) and by variation of the parameter k, we are able to change the location of the critical point of the order parameter equation. With critical point we denote hereby the one where the bifurcation from the stationary behavior to the periodic behavior occurs. This critical point is shifted in such a way, that the stationary states remain stable in the whole "working area". The order parameter equation (40) has then the following form:

$$\xi^u_{n+1} = (\lambda_u - k)\xi^u_n + \frac{a}{b(\lambda_s - \lambda_u)}(\lambda_s A_2(\xi^u_n)^2 + \lambda_u \xi^u_n)^2 + k\xi^u_{n-1}\,. \tag{43}$$

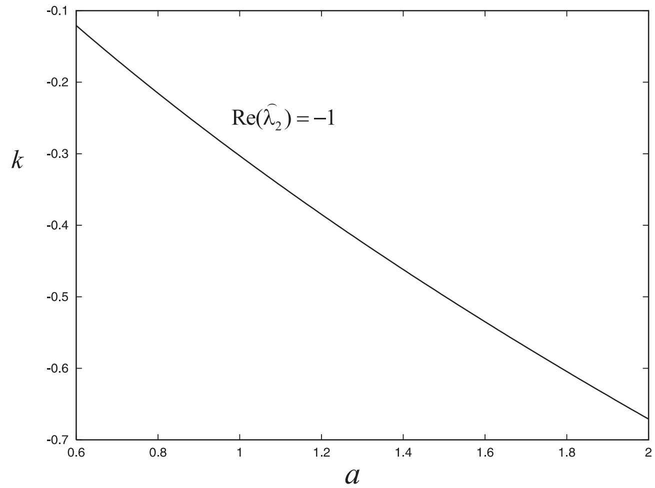

Fig. 4. Real part of the unstable eigenvalue $\hat\lambda_2$ as a function of the parameters a, k.

Obviously, this one-dimensional equation is equivalent to the two-dimensional system

$$\begin{aligned}\xi^u_{n+1} &= (\lambda_u - k)\xi^u_n \\ &\quad + \frac{a}{b(\lambda_s - \lambda_u)}(\lambda_s A_2(\xi^u_n)^2 + \lambda_u \xi^u_n)^2 + k\eta_n \\ \eta_{n+1} &= \xi^u_n\end{aligned} \tag{44}$$

with a new state variable η_n. In order to investigate the stability of Eqs. (44) we perform a linear stability analysis and yield the following eigenvalues

$$\hat\lambda_{1,2} = \frac{\lambda_u - k}{2} \pm \frac{1}{2}\sqrt{(\lambda_u - k)^2 + 4k} \tag{45}$$

and the matrix of the eigenvectors

$$\underline{\underline{\hat V}} = \begin{pmatrix} \hat\lambda_s & \hat\lambda_u \\ 1 & 1 \end{pmatrix}. \tag{46}$$

From the condition $|\hat\lambda_{1,2}| > 1$ at $a_{\mathrm{crit}} = a^n_{\mathrm{crit}}$ where a^n_{crit} is the new critical value of the control parameter a, we find the corresponding values of the parameter k. At this value the order parameter equation loses its stability. The dependence of the unstable eigenvalue $\hat\lambda_2$ from the parameters a and k is shown in Fig. 4. Numerical investigations show, that with this method it is possible to shift the bifurcation point over a wide range, even further than the second bifurcation of period-doubling section which occurs at $a_{\mathrm{crit}} = 0.9125$. To derive the order parameter equation for the system (44) with a quadratic center manifold we get the following mode equations:

$$\begin{aligned} \varphi^s_{n+1} &= \hat{\lambda}_s \varphi^s_n + \hat{K}\hat{\Phi} \\ \varphi^u_{n+1} &= \hat{\lambda}_u \varphi^u_n - \hat{K}\hat{\Phi} \end{aligned} \tag{47}$$

with

$$\hat{K} = \frac{a}{b(\lambda_s - \lambda_u)(\hat{\lambda}_s - \hat{\lambda}_u)} \tag{48}$$

$$\begin{aligned} \hat{\Phi} &= (\lambda_u + \hat{\lambda}_s \lambda_s A_2 \varphi^s_n + \hat{\lambda}_s \lambda_u A_2 \varphi^u_n)^2 \\ &\quad \times (\hat{\lambda}_s \varphi^s_n + \hat{\lambda}_u \varphi^u_n)^2 \end{aligned} \tag{49}$$

Here φ^s_n, φ^u_n are accordingly new stable and unstable modes. Now we apply again the center manifold theorem and restrict ourselves up to the second order with respect to the smallness parameter ε, which means:

$$\varphi^s_n = h(\varphi^u_n) = \hat{A}_2(\varphi^u_n)^2 + O(3)\,. \tag{50}$$

To proceed further we insert (50) in (47), and expand all ε-depended terms into a Taylor series with respect to ε. After that we neglect all terms of the order $O(\varepsilon^{\frac{3}{2}})$ keeping in mind that for the period-doubling bifurcation $\varphi^u_n \sim \varepsilon^{\frac{1}{2}}$ holds. As result we get

$$\begin{aligned} \varphi^s_{n+1} &= \hat{\lambda}_s \varphi^s_n + \hat{K}\hat{\Phi} \\ &= \hat{A}_2 \hat{\lambda}^0_s \varepsilon + (\hat{\lambda}^0_u)^2 \hat{K}^0 (\lambda^0_u)^2 \varepsilon + O(\varepsilon^{\frac{3}{2}}) \\ \varphi^u_{n+1} &= \hat{\lambda}_u \varphi^u_n - \hat{K}\hat{\Phi} \\ &= \hat{\lambda}^0_u \varepsilon^{\frac{1}{2}} - (\hat{\lambda}^0_u)^2 \hat{K}^0 (\lambda^0_u)^2 \varepsilon + O(\varepsilon^{\frac{3}{2}})\,, \end{aligned} \tag{51}$$

where $\hat{K}^0$ is the first term of expressions (48) in the Taylor series with respect to the smallness parameter ε. From the analogy to Eq. (7) it follows then

$$\hat{A}_2 \varepsilon (\hat{\lambda}^0_u)^2 = ((\hat{\lambda}^0_u)^2 \hat{K}^0 (\lambda^0_u)^2 + \hat{A}_2 \hat{\lambda}^0_s)\varepsilon\,, \tag{52}$$

and from that for the coefficient of the center manifold

$$\hat{A}_2 = \hat{K}^0 \frac{(\hat{\lambda}^0_u \lambda^0_u)^2}{(\hat{\lambda}^0_u)^2 - \hat{\lambda}^0_s}\,. \tag{53}$$

Finally the order parameter equation has then the following form:

$$\begin{aligned} \varphi^u_{n+1} = \hat{\lambda}_u \varphi^u_n - \hat{K}(\lambda_u &+ \hat{\lambda}_s \lambda_s A_2 \hat{A}_2 (\varphi^u_n)^2 \\ &+ \hat{\lambda}_s \lambda_u A_2 \varphi^u_n)^2 (\hat{\lambda}_s \hat{A}_2 (\varphi^u_n)^2 + \hat{\lambda}_u \varphi^u_n)^2\,. \end{aligned} \tag{54}$$

Numerical investigations of Eq. (54) show, that with this method it is indeed possible to shift the bifurcation point remarkably, whereas the type of the bifurcation, i.e. the period-doubling bifurcation remains unchanged.

3.2. *Controlling the mode equations*

As already mentioned we turn now to the treatment of the mode equations. In general the number of mode equations is larger than the number of order parameter equations and so the question arises in which of these equations the same control mechanism, found to be suitable, should be implemented. There exist at least two possibilities depending on the number of state variables: In the first alternative the control mechanism is implemented only in the unstable mode equation, whereas in the second one it is implemented in one or more of the stable mode equations as well as in the unstable mode equations. Controlling only the stable mode equations is not suitable because in this case the unstable mode equation remains unstable. This is a consequence of the fact, that in this case the unstable eigenvalues are not affected by the control mechanism.

Concerning our two-dimensional example of the Hénon map the first alternative leads to

$$\begin{aligned} \xi^s_{n+1} &= \lambda_s \xi^s_n + N_s(\xi^u_n,\, \xi^s_n,\, \{\alpha\}) \\ \xi^u_{n+1} &= \lambda_u \xi^u_n + N_u(\xi^u_n,\, \xi^s_n,\, \{\alpha\}) + F(k,\, \xi^u_n)\,, \end{aligned} \tag{55}$$

whereas in the second case we end up with

$$\begin{aligned} \xi^s_{n+1} &= \lambda_s \xi^s_n + N_s(\xi^u_n,\, \xi^s_n,\, \{\alpha\}) + F_1(k,\, \xi^s_n) \\ \xi^u_{n+1} &= \lambda_u \xi^u_n + N_u(\xi^u_n,\, \xi^s_n,\, \{\alpha\}) + F_2(k,\, \xi^u_n)\,. \end{aligned} \tag{56}$$

The different influence of Eqs. (55) and (56) on the original system is shown at the end of this section, but first we have to investigate whether the two alternatives lead to suitable behavior. In order to do this we apply again the order parameter concept. First we start with the system (55) with the inserted control function (41)

$$\begin{aligned} \xi^s_{n+1} &= \lambda_s \xi^s_n + N_s(\xi^u_n,\, \xi^s_n,\, \{\alpha\}) \\ \xi^u_{n+1} &= \lambda_u \xi^u_n + N_u(\xi^u_n,\, \xi^s_n,\, \{\alpha\}) + k(\xi^u_{n-1} - \xi^u_n)\,. \end{aligned} \tag{57}$$

Introducing a new state variable η_n we are able to get rid of the delay. Hence we yield the following three-dimensional system:

$$\begin{aligned}\xi^s_{n+1} &= \lambda_s \xi^s_n - K\Phi \\ \xi^u_{n+1} &= \lambda_u \xi^u_n + K\Phi + k(\eta_n - \xi^u_n) \\ \eta_{n+1} &= \xi^u_n\,,\end{aligned} \tag{58}$$

where K and Φ are defined by (35). The mode equations for (58) read

$$\begin{aligned}\varphi^{s_1}_{n+1} &= \tilde\lambda_{s_1}\varphi^{s_1}_n - K\tilde\Phi \\ \varphi^{s_2}_{n+1} &= \tilde\lambda_{s_2}\varphi^{s_2}_n + \tilde K\tilde\Phi \\ \varphi^{u}_{n+1} &= \tilde\lambda_{u}\varphi^{u}_n - \tilde K\tilde\Phi\end{aligned} \tag{59}$$

with the abbreviations

$$\begin{aligned}\tilde\Phi &= (\lambda_s\varphi^{s_1}_n + \lambda_u(\tilde\lambda_{s_2}\varphi^{s_2}_n + \tilde\lambda_u\varphi^u_n))^2 \\ \tilde K &= \frac{a}{b(\lambda_s-\lambda_u)(\tilde\lambda_{s_2}-\tilde\lambda_u)}\,.\end{aligned} \tag{60}$$

Hereby $\varphi^{s_1}_n$, $\varphi^{s_2}_n$, φ^u_n are the amplitudes and $\tilde\lambda_{s_1}$, $\tilde\lambda_{s_2}$, $\tilde\lambda_u$ the corresponding eigenvalues of the modes of the system (59). From the linear stability analysis we get for the eigenvalues

$$\tilde\lambda_{s_1} = \lambda_s,\ \tilde\lambda_{s_2} = \frac{\lambda_u - k + \sqrt{(\lambda_u-k)^2+4k}}{2}\,,$$

$$\tilde\lambda_u = \frac{\lambda_u - k - \sqrt{(\lambda_u-k)^2+4k}}{2}\,,$$

where λ_s, λ_u are the eigenvalues (12) of the system (23). The matrix of the eigenvectors has the following form:

$$\underline{\underline{\tilde V}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde\lambda_{s_2} & \tilde\lambda_u \\ 0 & 1 & 1\end{pmatrix}\,. \tag{61}$$

Our intention was to control the behavior of the system, in this case to shift the first period-doubling bifurcation, which occurs at the value $a_{\mathrm{crit}} = 147/400 = 0.3675$ via the insertion of the function (41). As example we shift the bifurcation point to the value $a^1_{\mathrm{crit}} = 1$. From the condition $|\tilde\lambda_u| = 1$ we yield $k = k^1_{\mathrm{crit}} = -0.302893182$. The eigenvalues $\tilde\lambda_{s_1}$, $\tilde\lambda_{s_2}$ remain stable, i.e. $|\tilde\lambda_{s_1}| < 1$, $|\tilde\lambda_{s_2}| < 1$.

For the two center manifolds in this case we use again

$$\varphi^{s_1}_n = h_1(\varphi^u_n) = \tilde A_2(\varphi^u_n)^2 + O(3)\,, \tag{62}$$

$$\varphi^{s_2}_n = h_2(\varphi^u_n) = \tilde B_2(\varphi^u_n)^2 + O(3)\,, \tag{63}$$

where we keep only terms up to the first order with respect to ε taking into account that $\varphi^u_n \sim \varepsilon^{\frac12}$. A numerical experiment has confirmed as expected, that indeed $\alpha = 1/2$ holds (see Table 4). With the expressions $\varphi^{s_1}_{n+1} = h_1(\varphi^u_{n+1})$ and $\varphi^{s_2}_{n+1} = h_2(\varphi^u_{n+1})$ one yields from Eq. (7)

$$\begin{aligned}\tilde A_2(\tilde\lambda^0_u)^2\varepsilon^{2\alpha} &= \tilde\lambda^0_{s_1}\tilde A_2\varepsilon^{2\alpha} - \frac{a_{\mathrm{crit}}(\lambda^0_u\tilde\lambda^0_u)^2\varepsilon^{2\alpha}}{b(\lambda^0_s-\lambda^0_u)} \\ \tilde B_2(\tilde\lambda^0_u)^2\varepsilon^{2\alpha} &= \tilde\lambda^0_{s_2}\tilde B_2\varepsilon^{2\alpha} + \frac{a_{\mathrm{crit}}(\lambda^0_u\tilde\lambda^0_u)^2\varepsilon^{2\alpha}}{b(\lambda^0_s-\lambda^0_u)(\tilde\lambda^0_{s_2}-\tilde\lambda^0_u)}\end{aligned} \tag{64}$$

and therefore for the coefficients of the center manifolds:

$$\tilde A_2 = K^0\frac{(\lambda^0_u\tilde\lambda^0_u)^2}{\tilde\lambda^0_{s_1} - (\tilde\lambda^0_u)^2}\,, \tag{65}$$

$$\tilde B_2 = -\tilde K^0\frac{(\lambda^0_u\tilde\lambda^0_u)^2}{\tilde\lambda^0_{s_2} - (\tilde\lambda^0_u)^2}\,. \tag{66}$$

Hereby K^0, $\tilde K^0$, $\tilde\lambda^0$ are the terms of lowest order in the Taylor series with respect to the smallness parameter ε of expressions (35) and correspondingly (60), and λ^0_u is exactly equal to $\lambda_u(a_{\mathrm{crit}})$, where $a^1_{\mathrm{crit}} = 1$. In Fig. 5 one can see both the dependence of the center manifolds $\varphi^{s_1}_n = h_1(\varphi^u_n)$ and $\varphi^{s_2}_n = h_2(\varphi^u_n)$ on the unstable mode φ^u_n as well as the corresponding projections of the numerical solution of the mode equations (59). Now we can derive the order parameter equation which has the following form:

$$\begin{aligned}\varphi^u_{n+1} = \tilde\lambda_u\varphi^u_n &- \frac{a}{b(\lambda_s-\lambda_u)(\tilde\lambda_{s_2}-\tilde\lambda_u)} \\ &\times(\lambda_s(\tilde A_2(\varphi^u_n)^2 + \lambda_u(\tilde\lambda_{s_2}\tilde B_2(\varphi^u_n)^2 + \tilde\lambda_u\varphi^u_n))^2,\end{aligned} \tag{67}$$

where $\tilde A_2$, $\tilde B_2$ are the coefficients (65) and (66). In order to verify our analytical result Fig. 6 shows the bifurcation diagram obtained numerically of both the order parameter equation as well as the mode equations. As one can see there is at least in the vicinity of the bifurcation at $a^1_{\mathrm{crit}} = 1$ a good coincidence of both solutions.

Now we consider the system (56) and pursue the same approach. In the first step we get then

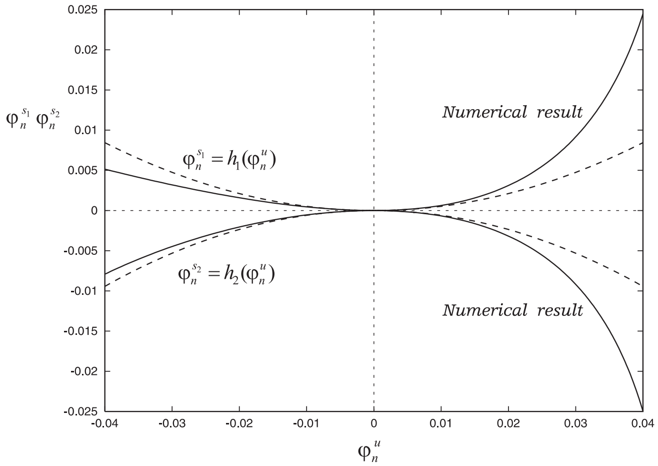

Fig. 5. Dependencies $\varphi_n^{s_1} = h_1(\varphi_n^u)$ and $\varphi_n^{s_1} = h_1(\varphi_n^u)$ obtained analytically (62), (63) and by simulation from the mode equations (59).

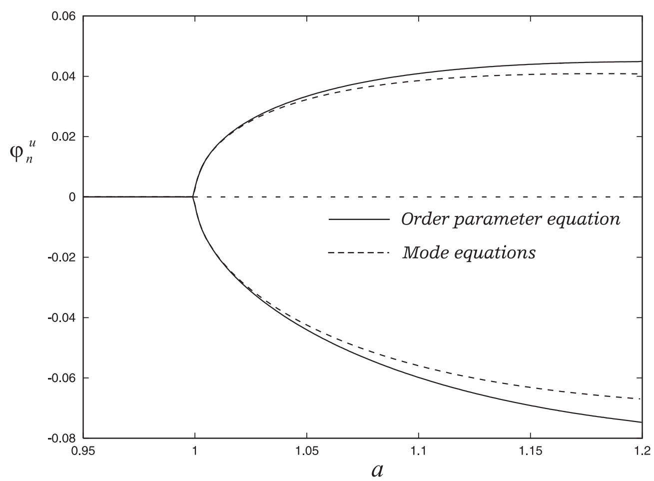

Fig. 6. Numerically obtained bifurcation diagram of the order parameter equation (67) and the mode equations (59).

$$\begin{aligned}\xi_{n+1}^s &= \lambda_s\xi_n^s + N_s(\xi_n^u,\, \xi_n^s,\, \{\alpha\}) + k(\xi_{n-1}^s - \xi_n^s)\\ \xi_{n+1}^u &= \lambda_u\xi_n^u + N_u(\xi_n^u,\, \xi_n^s,\, \{\alpha\}) + k(\xi_{n-1}^u - \xi_n^u)\,.\end{aligned} \tag{68}$$

The corresponding mode equations read in detail

$$\varphi_{n+1}^{s_1} = \check{\lambda}_{s_1}\varphi_n^{s_1} - \check{K}_1\check{\Phi} \tag{69a}$$

$$\varphi_{n+1}^{s_2} = \check{\lambda}_{s_2}\varphi_n^{s_2} + \check{K}_1\check{\Phi} \tag{69b}$$

$$\varphi_{n+1}^{s_3} = \check{\lambda}_{s_3}\varphi_n^{s_3} + \check{K}_2\check{\Phi} \tag{69c}$$

$$\varphi_{n+1}^{u} = \check{\lambda}_{u}\varphi_n^{u} - \check{K}_2\check{\Phi} \tag{69d}$$

with

$$\begin{aligned}\check{\Phi} &= (\lambda_s(\check{\lambda}_{s_1}\varphi_n^{s_1} + \check{\lambda}_{s_2}\varphi_n^{s_2}) + \lambda_u(\check{\lambda}_{s_3}\varphi_n^{s_3} + \check{\lambda}_u\varphi_n^u))^2\\ \check{K}_1 &= \frac{a}{b(\lambda_s - \lambda_u)(\check{\lambda}_{s_1} - \check{\lambda}_{s_2})}\,,\\ \check{K}_2 &= \frac{a}{b(\lambda_s - \lambda_u)(\check{\lambda}_{s_3} - \check{\lambda}_{u})}\,.\end{aligned} \tag{70}$$

The eigenvalues and the matrix of the eigenvectors of Eqs. (69) have the following form:

$$\begin{aligned}\check{\lambda}_{s_{1,2}} &= \frac{\lambda_s - k \pm \sqrt{(\lambda_s - k)^2 + 4k}}{2}\,,\\ \check{\lambda}_{s_3,u} &= \frac{\lambda_u - k \pm \sqrt{(\lambda_u - k)^2 + 4k}}{2}\end{aligned} \tag{71}$$

and

$$\underline{\underline{V}} = \begin{pmatrix} \check{\lambda}_{s_1} & \check{\lambda}_{s_2} & 0 & 0\\ 0 & 0 & \check{\lambda}_{s_3} & \check{\lambda}_u\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}. \tag{72}$$

From (71) and the corresponding values of λ_s and $k = k_{\text{crit}}^1$ it can be shown, that Eqs. (69a) and (69b) are complex conjugate. For the quadratic center manifolds

$$\begin{aligned}\varphi_n^{s_1} &= h_1(\varphi_n^u) = \check{A}_2(\varphi_n^u)^2 + O(3)\\ \varphi_n^{s_2} &= h_2(\varphi_n^u) = \check{B}_2(\varphi_n^u)^2 + O(3)\\ \varphi_n^{s_3} &= h_3(\varphi_n^u) = \check{C}_2(\varphi_n^u)^2 + O(3)\end{aligned} \tag{73}$$

we yield in this case from the analogy to Eq. (7)

$$\begin{aligned}\check{A}_2(\check{\lambda}_u^0)^2\varepsilon^{2\alpha} &= \check{\lambda}_{s_1}^0\check{A}_2\varepsilon^{2\alpha} - \varepsilon^{2\alpha}\frac{a_{\text{crit}}(\lambda_u^0\check{\lambda}_u^0)^2}{b(\lambda_s^0 - \lambda_u^0)(\check{\lambda}_{s_1}^0 - \check{\lambda}_{s_2}^0)}\\ \check{B}_2(\check{\lambda}_u^0)^2\varepsilon^{2\alpha} &= \check{\lambda}_{s_2}^0\check{B}_2\varepsilon^{2\alpha} + \varepsilon^{2\alpha}\frac{a_{\text{crit}}(\lambda_u^0\check{\lambda}_u^0)^2}{b(\lambda_s^0 - \lambda_u^0)(\check{\lambda}_{s_1}^0 - \check{\lambda}_{s_2}^0)}\\ \check{C}_2(\check{\lambda}_u^0)^2\varepsilon^{2\alpha} &= \check{\lambda}_{s_3}^0\check{C}_2\varepsilon^{2\alpha} + \varepsilon^{2\alpha}\frac{a_{\text{crit}}(\lambda_u^0\check{\lambda}_u^0)^2}{b(\lambda_s^0 - \lambda_u^0)(\check{\lambda}_{s_3}^0 - \check{\lambda}_{u}^0)}\end{aligned} \tag{74}$$

with the following coefficients $\check{A}_2$, $\check{B}_2$, $\check{C}_2$:

$$\check{A}_2 = \check{K}_1^0\frac{(\lambda_u^0\check{\lambda}_u^0)^2}{\check{\lambda}_{s_1}^0 - (\check{\lambda}_u^0)^2}\,, \tag{75}$$

$$\check{B}_2 = \check{K}_1^0\frac{(\lambda_u^0\check{\lambda}_u^0)^2}{(\check{\lambda}_u^0)^2 - \check{\lambda}_{s_2}^0}\,, \tag{76}$$

$$\check{C}_2 = \check{K}_2^0\frac{(\lambda_u^0\check{\lambda}_u^0)^2}{(\check{\lambda}_u^0)^2 - \check{\lambda}_{s_3}^0}\,, \tag{77}$$

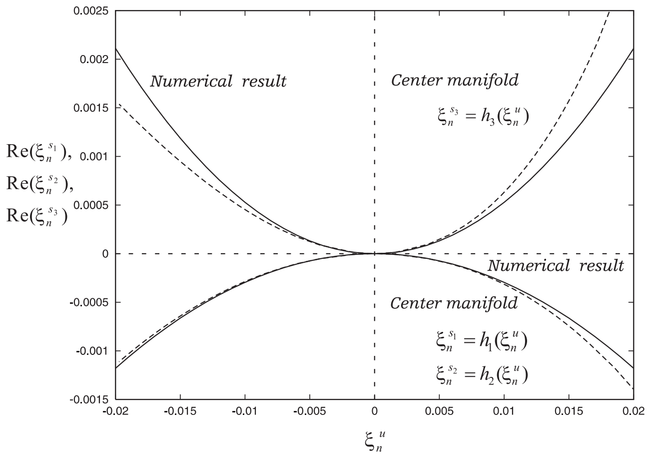

Fig. 7. Real part of the quadratic center manifold (75)–(77) together with the result obtained by a numerical simulation from the mode equations (69).

where $\check{K}_1^0$, $\check{K}_2^0$ are the terms of lowest order in the Taylor series with respect to the smallness parameter ε of expression (70).

Below are shown the numerical values of these coefficients for the values of the parameters $k_{\text{crit}}^1 = -0.302893182$, $a_{\text{crit}}^1 = 1$, $b = 0.3$

$$\check{A}_2 = -2.948179547 + I4.516842572\,,$$
$$\check{B}_2 = -2.948179547 - I4.516842572\,,$$
$$\check{C}_2 = 5.279103128\,.$$

Substituting the obtained results in system (69) we obtain the order parameter equation

$$\varphi_{n+1}^u = \check{\lambda}_u\varphi_n^u - \frac{a}{b(\lambda_s-\lambda_u)(\check{\lambda}_{s_3}-\check{\lambda}_u)}(\lambda_s(\check{\lambda}_{s_1}\check{A}_2(\varphi_n^u)^2 + \check{\lambda}_{s_2}\check{B}_2(\varphi_n^u)^2) + \lambda_u(\check{\lambda}_{s_3}\check{C}_2(\varphi_n^u)^2 + \check{\lambda}_u\varphi_n^u))^2 \quad (78)$$

In Figs. 7 and 8 the real and imaginary parts of the center manifolds are shown together with the projections of the numerically simulated mode equations (69). Figure 9 finally shows the bifurcation diagram of the order parameter equation and the numerically simulated mode equations.

For Eqs. (43), (57), (68) and (86) we calculate numerically in Table 4 the coefficient α [see (33)] at the bifurcation points for various values of the parameter k in order to verify the assumption (33). The calculations were made with 500 000 iterations. In the intervals of the parameter a one hundred logarithmically equidistant points are chosen and the values of α were obtained by a least squares fit assuming the following functional dependence $\log(\xi_n^u) = \alpha * \log(\varepsilon) + \beta$ (respectively, $\log(x_n - x_{st}) = \alpha * \log(\varepsilon) + \beta$ for the original Hénon map). In Table 4 the upper values correspond to the upper branch of the bifurcation and the lower values to the lower branch of the bifurcation.

To summarize our results, we can conclude that the stabilization of the unstable mode is important for this control mechanism in order to obtain the suitable behavior of the system. In Table 5 we have collected the coefficients of the order parameter equations (54), (67) and (78). As one can see they coincide in the first too lowest orders but differ in the higher orders which is a consequence

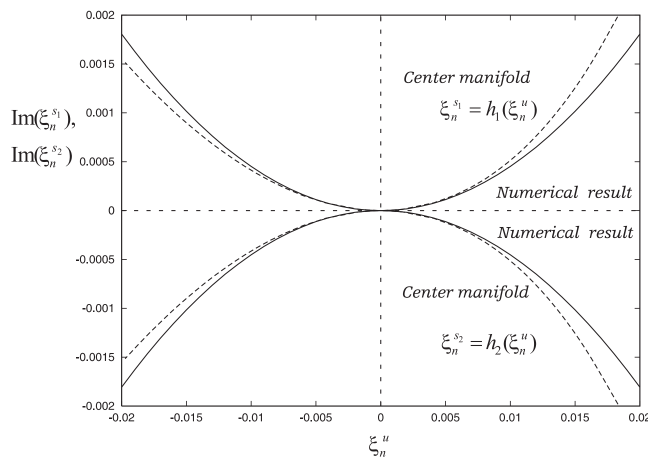

Fig. 8. Imaginary part of the quadratic center manifold (75)–(77) together with the result obtained by a numerical simulation from the mode equations (69).

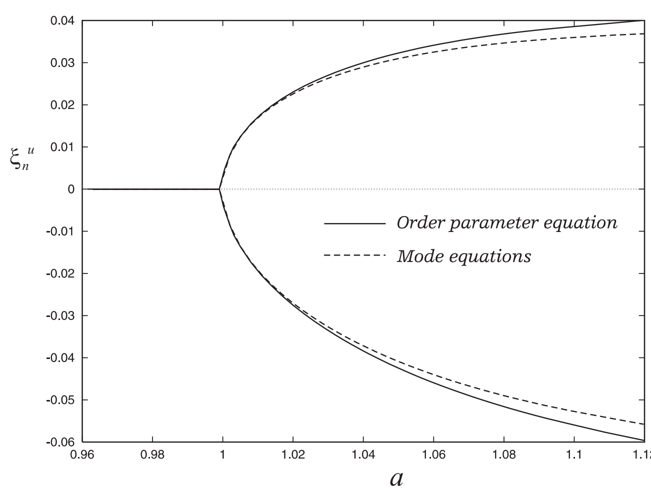

Fig. 9. Numerically obtained bifurcation diagram of the order parameter equation (78) (dimension 1) and the mode equations (69) (dimension 4).

Table 5. The coefficients of the order parameter equations (54), (67) and (78).

Terms	Order Param. Eq. (54)	Order Param. Eq. (67)	Order Param. Eq. (78)
φ_n^u	$\overline{\lambda}$	$\overline{\lambda}$	$\overline{\lambda}$
$(\varphi_n^u)^2$	$K_e\lambda_u\overline{\lambda}$	$K_e\lambda_u\overline{\lambda}$	$K_e\lambda_u\overline{\lambda}$
$(\varphi_n^u)^3$	$2K_e(\lambda_u\overline{\overline{\lambda}}\hat{A}_2+\lambda_s\overline{\lambda}^2A_2)$	$2K_e(\lambda_u\overline{\overline{\lambda}}\tilde{B}_2+\lambda_s\tilde{A}_2)$	$2K_e(\lambda_u\overline{\overline{\lambda}}\check{C}_2+\lambda_s(\check{\lambda}_{s_1}\check{A}_2+\check{\lambda}_{s_2}\check{B}_2)$
$(\varphi_n^u)^4$	$\dfrac{K_e((\lambda_u\overline{\overline{\lambda}}\hat{A}_2+\lambda_s\overline{\lambda}^2A_2)^2+K_d)}{\lambda_u\overline{\lambda}}$	$\dfrac{K_e(\lambda_u\overline{\overline{\lambda}}\tilde{B}_2+\lambda_sA_2)^2}{\lambda_u\overline{\lambda}}$	$\dfrac{K_e(\lambda_u\overline{\overline{\lambda}}\check{C}_2+\lambda_s(\check{\lambda}_{s_1}\check{A}_2+\check{\lambda}_{s_2}\check{B}_2)^2}{\lambda_u\overline{\lambda}}$

$$\overline{\lambda}=\tilde{\lambda}_u=\hat{\lambda}_u=\check{\lambda}_u=\frac{\lambda_u-k-\sqrt{(\lambda_u-k)^2+4k}}{2},\quad \overline{\overline{\lambda}}=\tilde{\lambda}_{s_2}=\hat{\lambda}_s=\check{\lambda}_{s_3}=\frac{\lambda_u-k+\sqrt{(\lambda_u-k)^2+4k}}{2}$$

$$K_e=-\frac{a\lambda_u\overline{\lambda}}{b(\lambda_s-\lambda_u)\sqrt{(\lambda_u-k)^2+4k}},\ K_d=4\lambda_u\overline{\overline{\lambda}}\hat{A}_2\lambda_sA_2\overline{\lambda}^2$$

of the dimensionality of the corresponding original system.

3.3. *Transition from the mode equations to the initial equations*

Although we have applied the control mechanism to the mode equations we are mainly interested in the effect of the control mechanism on the original system, therefore we consider now the transition from the modes equations (57), (68) back to the original equations. Starting from Eq. (19) and inserting the control mechanism we get

$$\begin{aligned}\underline{\xi}_{n+1}&=\Gamma^{\Lambda}_{(2)}(:\underline{\xi}_n)+\Gamma^{\tilde{N}}_{(3)}(:\underline{\xi}_n)^2\\&\quad+k\Gamma^{S}_{(2)}(:\underline{\xi}_{n-1}-\underline{\xi}_n)\,,\end{aligned}\tag{79}$$

where we have introduced the tensor $\Gamma^S_{(2)}$. The components of this tensor are shown for Eq. (57) in Table 6 and for Eq. (68) in Table 7. Multiplying (79) from the left side with the tensor $\Gamma^V_{(2)}$ and obeying (20) and (21) from Sec. 2 we derive

$$\begin{aligned}\Gamma^V_{(2)}(\underline{\xi}_{n+1})&=\Gamma^L_{(2)}\Gamma^V_{(2)}(:\underline{\xi}_n)+\Gamma^N_{(3)}(\Gamma^V_{(2)}(:\underline{\xi}_n))^2\\&\quad+[k\Gamma^V_{(2)}\Gamma^S_{(2)}\Gamma^{V^{-1}}_{(2)}(\Gamma^V_{(2)}(:\underline{\xi}_{n-1}-\underline{\xi}_n))].\end{aligned}\tag{80}$$

The Eq. (80) is analogue to Eq. (18) of Sec. 2 but contains in addition the expression in square brackets which is caused by the control mechanism. If we add the stable stationary state $\underline{q}_{st}$ to Eq. (80) we finally yield

$$\underline{x}_{n+1}=\underline{N}(\underline{x}_n,\{\alpha\})+[k\Gamma^V_{(2)}\Gamma^S_{(2)}\Gamma^{V^{-1}}_{(2)}(:\underline{x}_{n-1}-\underline{x}_n)]\tag{81}$$

with

$$\underline{N}(\underline{x}_n,\{\alpha\})=\begin{pmatrix}1+y_n-ax_n^2\\bx_n\end{pmatrix}.\tag{82}$$

From Eq. (81) and Table 6, i.e. the control scheme of Eq. (57), we get the following matrix of control coefficients for the product of the tensors:

$$k\Gamma^V_{(2)}\Gamma^S_{(2)}\Gamma^{V^{-1}}_{(2)}=k\begin{pmatrix}A&B\\C&D\end{pmatrix},\tag{83}$$

where the coefficients A,B,C,D in general are different and depend on the control parameter k and the stationary state $\underline{q}_{st}$. With this, the components

Table 6. Components of tensor $\Gamma^S_{(2)}$ for Eq. (57).

$\Gamma^S_{(2)}$
$(\Gamma^S_{(2)})_{11}=1$
The remaining components are equal to zero

Table 7. Components of tensor $\Gamma^S_{(2)}$ for Eq. (68).

$\Gamma^S_{(2)}$
$(\Gamma^S_{(2)})_{11}=1$
$(\Gamma^S_{(2)})_{22}=1$
The remaining components are equal to zero

Table 8. Coefficients of the control (84) and (86).

Critical	Points	Parameter of the Control (86)	Parameter of the Extended Control (84)				
a_{crit}	a^n_{crit}	k	k	A	B	C	D
	1	-0.30289	-0.30289	0.89578	-0.55784	-0.16735	0.10421
0.3675	1.4	-0.46186	-0.46186	0.92501	-0.48084	-0.14425	0.07498
	1.6	-0.53491	-0.53491	0.93455	-0.45151	-0.13545	0.06544

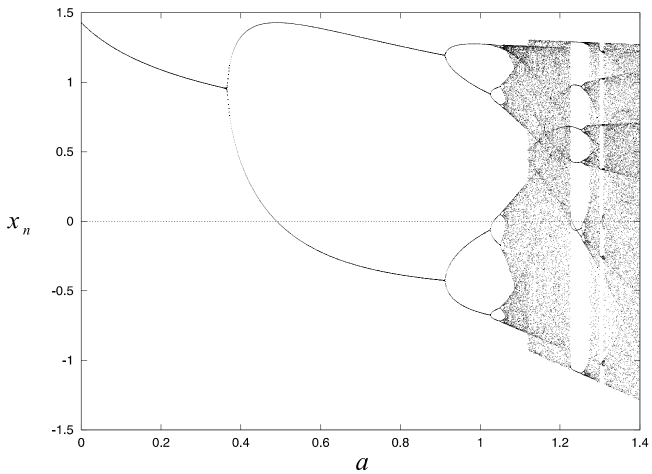

Fig. 10. The bifurcation diagram of the original Hénon map ($x(0) = 0.5$), Eq. (9).

of Eq. (81) will have the following form:

$$\begin{aligned} x_{n+1} &= 1 + y_n - ax_n^2 + k(A(x_{n-1} - x_n) \\ &\quad + B(y_{n-1} - y_n)) \\ y_{n+1} &= bx_n + k(C(x_{n-1} - x_n) + D(y_{n-1} - y_n)) \end{aligned} \tag{84}$$

For the second control scheme, i.e. Eq. (68) we obtain from Eq. (81) and Table 7

$$k\Gamma^V_{(2)}\Gamma^S_{(2)}\Gamma^{V^{-1}}_{(2)} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{85}$$

and for the components of Eq. (81)

$$\begin{aligned} x_{n+1} &= 1 + y_n - ax_n^2 + k(x_{n-1} - x_n) \\ y_{n+1} &= bx_n + k(y_{n-1} - y_n)\,. \end{aligned} \tag{86}$$

Looking at Eqs. (84) and (86) it is interesting to remark, that the more complex feedback mechanism corresponding to Eq. (68) leads to the simpler system (86). Numerical investigations of both systems show however, that this control mechanism leads to a behavior which is generally more stable. We assume that the reason for that lies in the

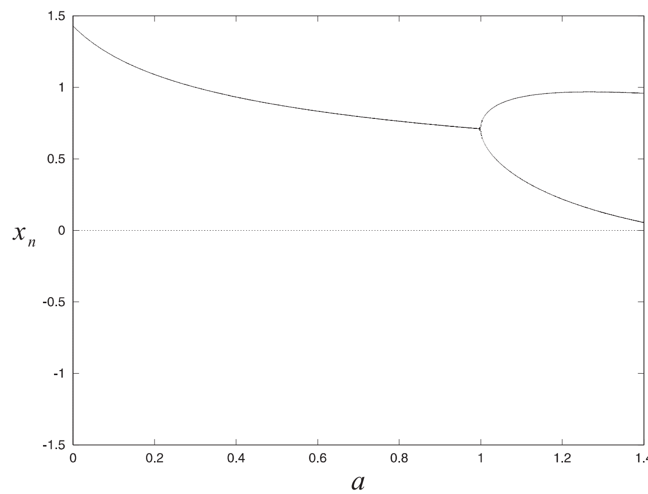

Fig. 11. Shifted value of the first bifurcation $a^n_{\mathrm{crit}} = 1$ ($x(0) = 0.5$), Eqs. (84) and (86).

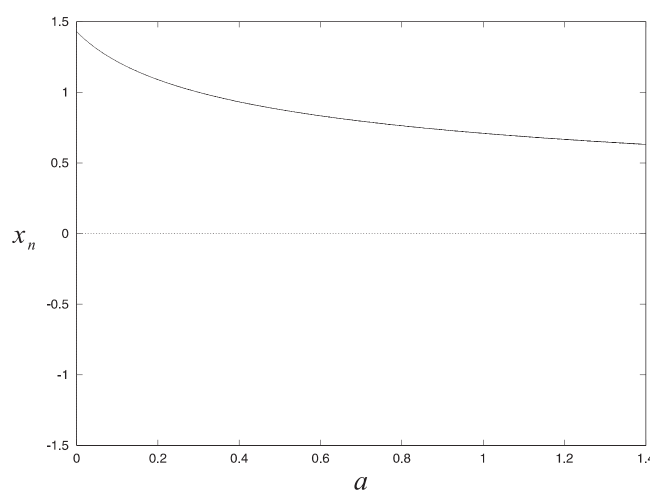

Fig. 12. The bifurcation diagram of completely stabilized Hénon map $a^n_{\mathrm{crit}} = 1.6$ ($x(0) = 0.5$), Eqs. (84) and (86).

stabilization of both modes, the unstable as well as the stable one. In Table 8 we have collected the coefficients for both control mechanisms. Here $a_{\rm crit}$ represents the old value of the bifurcation point, whereas $a^n_{\rm crit}$ represents the new one.

In Fig. 10 the original bifurcation diagram of the Hénon-map is shown, whereas in Figs. 11 and 12 the bifurcation diagrams for the cases $a^n_{\rm crit} = 1$ and $a^n_{\rm crit} = 1.6$ for both control mechanisms are presented. As we can see it is possible to shift the first bifurcation point of the period-doubling cascade with the applied control mechanism.

4. Summary and Outlook

It is shown in this work, that the synergetic approach [Haken, 1983] based on the derivation and analysis of the order parameter equation using the center manifold theory, allows the description of the behavior of nonlinear time discrete dynamical systems. This concept is based on the fact that in the vicinity of an instability, i.e., a bifurcation only a few degrees of freedom govern the dynamical behavior of the system. The application of this approach to partial differential equations (PDE) [Bestehorn & Haken, 1990] or delay differential equations (DDE) [Wischert *et al.*, 1994], i.e. infinite-dimensional systems is very useful, because in this case the reduction of the systems complexity, i.e. the degrees of freedom is enormous. If one looks at the periodic spatiotemporal patterns occuring in coupled map systems, it is expected that in the vicinity of the bifurcation points a reduction of the systems complexity occurs in this case too.

However also the application to low-dimensional nonlinear dynamical systems discrete in time is encouraging, although the reduction is in this case not so remarkable. These investigations are also interesting for theoretical purposes to develop suitable control mechanisms which can be applied to higher-dimensional systems up to infinite-dimensional systems like PDEs or DDEs. Therefore we have investigated the influence of well-known control mechanisms (see [Pyragas, 1993]) on the order parameter equation, the mode equations and finally the original evolution equation. The calculations in this work show, that it is possible to derive the order parameter equations for nonlinear maps with dimensionalities 2, 3 and 4 analytically. From the point of view of control theory, we have shown the efficiency of the stabilization by shifting the bifurcation point using the method of delayed feedback mentioned above and in addition derived the modified control scheme (see also [Basso *et al.*, 1998]). The advantage of this method is that we are able to calculate the appropriate value of the parameter $k_{\rm crit}$ for the specific bifurcation point $a^n_{\rm crit}$. Furthermore it can be formalized using a symbolic manipulation program like Maple which makes it possible to derive order parameter equations for systems with dimension four or even higher. However there are still some open questions which should be investigated further.

Concerning our assumption (33) for the dependence of the unstable mode amplitude ξ^u_n on the smallness parameter ε the question arises whether this functional dependence is fulfilled in all cases. Concerning the control of periodic orbits it has to be investigated which feedback schemes should be applied here in order to shift only the second and higher bifurcations, for instance, in a period-doubling scenario.

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